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EDITED BY

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CONTENTS.

JULY, 1859.

	PAGE
PRIZE PROBLEMS FOR STUDENTS,	321
REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN	
No. VI. VOL. I.,	322
NOTE ON DERIVATIVES,	326
NOTES ON THE THEORY OF PROBABILITIES. By Simon Newcomb, . . .	331
NOTE ON MAXIMA AND MINIMA. By Lewis R. Gibbs,	335
ARCS OF GREAT AND SMALL CIRCLES. By George P. Bond,	342
ON MR. COLLINS'S PROPERTY OF CIRCULATES. By James Edward Oliver, .	345
SOLUTIONS OF PROBLEMS IN PROBABILITIES. By Simon Newcomb, . . .	349
MATHEMATICAL MONTHLY NOTICES,	350
EDITORIAL ITEMS,	352

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THE
MATHEMATICAL MONTHLY.

VOL. I...JULY, 1859....No. X.

PRIZE PROBLEMS FOR STUDENTS.

I.

Solve the two equations

$$\begin{aligned}x^2 - 50x + xy + x^2y^5 + xy^6 &= 50y \\ x^2y - 100x + xy^2(1+y^2)(1+x) + xy^6 &= 100y.\end{aligned}$$

II.

If A, B, C be the angles, and a, b, c the opposite sides, in a plane triangle, of which S denotes the surface; prove that

$$a^2 + b^2 + c^2 = 4S(\cot A + \cot B + \cot C).$$

III.

If one of the similar triangles ABC and $A'B'C'$ be inscribed in the triangle DEF , and the other circumscribed about it; prove that the area of DEF will be a mean proportional between the areas of ABC and $A'B'C'$.

IV.

If a be one of the sides of an equilateral spherical triangle and A one of its angles, prove that $\sec A = \sec a + 1$.

V.

If the semiaxes of an ellipse be A and B , p the length of the

perpendicular dropped from the centre on the tangent to the curve, r and r' the distances from the point of tangency to the foci, and ρ the radius of curvature at this point; prove that

$$\rho = \frac{A^2 B^2}{p^3} = \frac{r r'}{p};$$

and from this theorem construct the corresponding point of the evolute.

The solution of these problems must be received by the first of September, 1859.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VI., Vol. I.

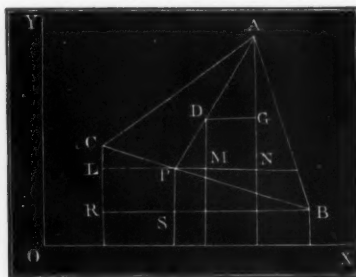
The first Prize is awarded to GEORGE B. HICKS, of Cleveland, Ohio.

The second Prize is awarded to ASHER B. EVANS, of the Junior Class in Madison University, Hamilton, N. Y.

PRIZE SOLUTION OF PROBLEM I.

"Any side of a triangle is cut in the ratio of m to n , and the line joining this point to the opposite vertex is cut in the ratio of $m+n$ to l ; to find the coördinates of the point of section."

Let ABC be any triangle, and let the coördinates of the vertices A, B, C be $x' y', x'' y'', x''' y'''$. Let the side BC be cut at



P in the ratio of m to n , and let xy be the coördinates of P . The triangles BPS and $PC L$ give

$$BP:PC::BS:LP; \text{ or, } n:m::x'-x:x-x''$$

$$BP:PC::PS:CL; \text{ or, } n:m::y-y':y''-y.$$

$$\therefore x = \frac{m x' + n x''}{m + n}, \quad y = \frac{m y' + n y''}{m + n}.$$

Again, let AP be cut at D in the ratio of $m+n$ to l , and let

α, β denote the coördinates of D . The triangles ADC and DPM give

$$AD:DP::DG:PM; \text{ or, } m+n:l::x'-\alpha:\alpha-\frac{mx''+nx'''}{m+n},$$

$$AD:DP::AG:DM; \text{ or, } m+n:l::y'-\beta:\beta-\frac{my''+ny'''}{m+n}.$$

$$\text{Therefore, } \alpha = \frac{lx'+mx''+nx'''}{l+m+n}, \beta = \frac{ly'+my''+ny'''}{l+m+n},$$

are the required coördinates.

This solution is by GEORGE B. HICKS.

PRIZE SOLUTION OF PROBLEM III.

"Find the polar equation of the line passing through the points, of which the polar coördinates are $r', \varphi'; r'', \varphi''$."

The equation of the straight line referred to rectangular coördinates, and passing through the points $x' y'$, and $x'' y''$, is

$$(1) \quad y-y' = \frac{y'-y''}{x'-x''}(x-x').$$

If the pole be at the origin, and the axis of x be the polar axis, then $x=r \cos \varphi$ and $y=r \sin \varphi$. Accenting, making the substitutions and reducing, (1) becomes

$$r r' (\sin \varphi \cos \varphi' - \cos \varphi \sin \varphi') + r r'' (\sin \varphi'' \cos \varphi - \cos \varphi'' \sin \varphi) + r' r'' (\sin \varphi' \cos \varphi'' - \cos \varphi' \sin \varphi'') = 0;$$

or, $r r' \sin (\varphi - \varphi') + r r'' \sin (\varphi'' - \varphi) + r' r'' \sin (\varphi' - \varphi'') = 0$, which is the equation required. This solution is by GEORGE B. HICKS.

PRIZE SOLUTION OF PROBLEM IV.

"Find the condition that $Ax+By+C=0$ should be tangent to $(x-a)^2+(y-b)^2=r^2$."

Let y be eliminated between the two equations, and the result arranged according to powers of x ; and we shall have

$$(A^2+B^2)x^2+2(AC+bAB-aB^2)x+B^2(a^2+b^2)+2bBC+C^2-r^2B^2=0.$$

In order that $Ax+By+C=0$ may be tangent to the circle, the

values of x from this quadratic must be equal. Whence, by the theory of equal roots,

$$(A C + b A B - a B^2)^2 = (A^2 + B^2) (a^2 B^2 + b^2 B^2 + 2 b B C + C^2 - r^2 B^2).$$

Developing and reducing, this equation becomes

$$r^2 (A^2 + B^2) = (A a + B b + C)^2,$$

which is the required condition.*

The condition may be verified as follows: It may be written $r = \pm \frac{A a + B b + C}{\sqrt{A^2 + B^2}}$. Now, we know by a simple proposition in analytical geometry, that the right-hand member of this equation expresses the length of the perpendicular from the point a, b upon the line $A x + B y + C = 0$; and since this perpendicular is equal to the radius of the circle, the coördinates of whose centre is a, b , the condition of tangency is fulfilled, as is known from plane geometry. It may be remarked also, that this condition gives a very concise solution of the problem. This solution is by GEORGE B. HICKS.

JOSEPH WINLOCK.

CHAUNCEY WRIGHT.

TRUMAN HENRY SAFFORD.

* The method of determining the equation of a tangent used here is the invention of DESCARTES. It is not confined in its application to curves of the second degree, but is generally applicable to all curves. Let the equation of any curve be $F(x, y) = 0$, and between this and the equation of the right line eliminate either variable, as y . The resulting equation will involve only x , the roots of which will be the values of x for the points where the line intersects the curve. If two of these points unite, two of the roots will be equal, and the line will become a tangent. The method by which DESCARTES determined the condition under which two of the roots would be equal was by assuming an equation of the same degree having two equal roots, and comparing it with the resulting equation. When the given curve is of the second degree, this ingenious artifice is rendered unnecessary, the solution of the equation being sufficient; as it is only necessary to put the radical term of the roots equal to zero.

This method of determining the equation of a tangent is that which appears in the letters of DESCARTES. That which he gives in his Geometry is somewhat different, and nearly as follows: Let

$$y^2 + (x - x_0)^2 - r^2 = 0$$

be the equation of a circle, the centre of which is on the axis of x . Let y be eliminated by means of this equation and that of the curve, and the roots of the resulting equation will be the values of x for the points where the circle meets the curve. The centre of the circle being supposed fixed and the radius arbitrary, let it be supposed to have such a value as will render two of the roots of the equation equal; the circle will then touch the curve, and therefore have the same rectilinear tangent. The value of r which renders the roots equal may be found by the artifice mentioned above. These methods are both founded on the same principle; and though we cannot but admire the ingenuity they display, yet they must in general yield to the more simple and direct method furnished by the Calculus.

Both of the above methods were used in the Prize solutions in the June number of the MONTHLY. See solutions of Problems I. and III.

The method of drawing tangents to curves, founded on the principles of the Differential Calculus, has superseded the other solutions for the same problem given by DESCARTES, FERMAT, ROBERVAL, and others. The methods given by these geometers were either limited to particular classes of curves, or in some cases so inconvenient as to amount nearly to impracticability. The determination of the equation of a tangent by the calculus is at once simple and general. It depends merely on differentiating the equation of the curve to find $\frac{dy}{dx}$, the tangent of the angle which the tangent line makes with the axis of x , and therefore extends to every curve capable of being expressed by an equation, and whose equation is capable of differentiation. The methods of DESCARTES, explained above, extend at most only to algebraic curves.

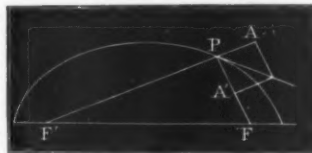
The method of ROBERVAL deserves notice, as well on account of the elegance of the conception on which it is founded, as of its close analogy to the fundamental principle of the Newtonian fluxions. He considered a curve described by a point affected with two motions, the variations of which in quantity and direction are to be determined by the nature of the curve. At any point of the curve he supposed a parallelogram constructed, the sides of which are proportional to and in the direction of the generating velocities, and laid it down as a principle, that the diagonal which represents the direction of the resultant is the direction of the element of the curve at that point, and therefore the direction of the tangent. There are many instances in which this method may be applied with great clearness and facility; but in most cases its application is either totally impracticable, or attended with very perplexing difficulties, owing to the intricacy of the investigations necessary to determine the component velocities of the generating points. We shall give some examples in which its application is effected with great clearness and beauty.

1. To determine the tangent to a point in an ellipse or hyperbola.

In the ellipse the sum of the distances $F'P$ and FP of the describing point from the foci is invariable; therefore one increases with the same velocity as the other diminishes. Hence the velocity of the describing point in the directions PA and PA' are equal; therefore if $PA = PA'$, the diagonal is the tangent which bisects the angle APA' .

In the hyperbola the difference of the distances from the foci is constant, and therefore the two distances increase with the same velocity. Hence PA' should in this case be taken on the produced part of the focal distance, as well as PA , and therefore the tangent bisects the angle under the radii vectores from the foci.

2. To draw a tangent to a given point in a parabola.

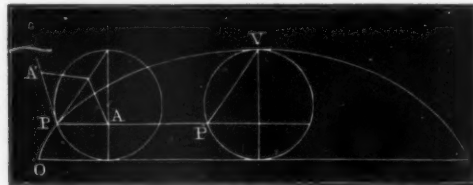


Let OB be the directrix, and OX the axis, and F the focus. By the properties of this curve, $FP = BP$. \therefore the velocities in the directions PA and PA' are equal; \therefore as before, the tangent bisects the angle APA' .



3. To draw a tangent at a given point in a cycloid.

Let P be the generating point. By the definition of the cycloid, the generating point at P has two motions, one in the direction of the tangent PA' to the generating circle, and the other in the direction PA parallel to the base; and these two motions are equal, because the generating point moves uniformly round the circumference of the generating circle in the same time that the circle itself is carried along the base through an equal space. Hence if PA and PA' represent the two motions $PA = PA'$, and therefore the tangent bisects the angle APA' , and is parallel to the corresponding chord $P'V$ of the generating circle described upon the axis.



This method of ROBERVAL is peculiarly applicable to curves which can be described mechanically by motion. BARROW subsequently invented a method of tangents which approached as near the principle of the differential calculus as ROBERVAL's did to the fluxional principle. He investigates an infinitely small triangle composed of the increments of the abscissa and ordinate, and the elementary arc of the curve. The student will readily perceive this to be the spirit of the differential calculus; but both this and the method of ROBERVAL want, what constitutes the principal excellence of the methods of the fluxional and differential calculus, that uniform algorithm by which a general formula expresses the equation of a tangent to any curve, and the general rules by which the particular values of the quantities composing this general formula can be found in particular cases. It should be observed, that the method of BARROW is very nearly the same as that of FERMAT. This note is taken from LARDNER's Algebraic Geometry. — Ed.

NOTE ON DERIVATIVES.

It seems to us, that the method of demonstrating the rules for finding the derivatives of many algebraic functions is not only most concise, but most easily understood by the learner, when based upon the following

PROPOSITION.* When i is an infinitesimal, $\text{Nap. log } (1 + i) = i$. By the binominal theorem,

* See Prof. PEIRCE's *Curves and Functions*, Vol. I.; also Prof. PRICE's *Infinitesimal Calculus*, Vol. I.

$$(1+i)^{\frac{1}{i}} = 1 + \frac{1}{i} \cdot i + \frac{1}{i} \left(\frac{1}{i} - 1 \right) \frac{i^2}{1 \cdot 2} + \frac{1}{i} \left(\frac{1}{i} - 1 \right) \left(\frac{1}{i} - 2 \right) \frac{i^3}{1 \cdot 2 \cdot 3} + \&c.$$

Since $\frac{1}{i}$ is infinite, and since the finite quantities which are added to or subtracted from infinite ones do not affect their values, $\frac{1}{i} = \frac{1}{i} - 1 = \frac{1}{i} - 2 = \&c.$, which reduces the development to

$$(1+i)^{\frac{1}{i}} = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

But this series, by reducing, and adding a sufficient number of its terms together, gives 2.7182818+, known as e , the base of NAPIER'S system of logarithms. Hence $(1+i)^{\frac{1}{i}} = e$; and

$$\frac{1}{i} \text{ Nap. log } (1+i) = \text{Nap. log } e = 1. \therefore \log (1+i) = i,$$

understanding that log denotes Nap. log.

We will now apply this proposition to finding the derivatives of a few functions, including Problem V., No. VI., Vol. I.

1. Find the derivative of $u = x^n$. $\log u = n \log x$. Give x an infinitesimal increment dx , and let du be the corresponding increment of u ; then $\log(u+du) = n \log(x+dx)$. The logarithm of the ratio of $u+du$ to u is

$$\log(u+du) - \log u = n \log(x+dx) - n \log x;$$

$$\text{or,} \quad \log \left(1 + \frac{du}{u} \right) = n \log \left(1 + \frac{dx}{x} \right).$$

Therefore, since $\frac{du}{u}$ and $\frac{dx}{x}$ are infinitesimals, we have by our proposition,

$$\log \left(1 + \frac{du}{u} \right) = \frac{du}{u} = n \log \left(1 + \frac{dx}{x} \right) = \frac{n dx}{x}.$$

$$\text{Therefore,} \quad \frac{du}{dx} = \frac{nu}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

In this case u is called a function of x ; that is, u depends upon x for its value, and varies with it. But it is evident, that u and x do not vary by the same amount; and it is the aim of the Differential Calculus to find the *ratio* of these variations when they are infini-

tesimals. This ratio, or quotient, is called by another name in the calculus; namely, *derivative* or *differential coefficient*. We started with the function x^n , and derived nx^{n-1} from it, as the ratio of the variation of the function to its variable x ; and hence nx^{n-1} is called the *derivative*, or *derived function* of x^n . Again, $du = nx^{n-1}dx$, in which nx^{n-1} is the *coefficient* of the differential dx ; and hence the name, *differential coefficient*. The student will observe, that $\frac{du}{dx}$ denotes the derivative of the quantity u ; but the symbol, as separated from the quantity, and simply denoting the operation, is $\frac{d}{dx}$. Thus, $\frac{d}{dx}f(x)$ tells us to find the derivative of $f(x)$. The inconvenience of the use of the symbol $\frac{d}{dx}$, in this and like cases has led to the adoption of D in its place. If we wish to indicate at the same time the particular variable, as x , in reference to which the derivative is to be taken, then the symbol D_x is used. Hence in symbols $\frac{d}{dx} = D_x$. When the function involves only a single variable, as x , D is sufficient; but in the symbol $\frac{d}{dx}$, the x cannot be omitted. In the case of a general function, as $f(x)$, the notation $f'(x)$, $f''(x)$, $f'''(x)$, &c., to denote the successive derivatives, was used by LAGRANGE, and is most convenient.

So far as we know, Prof. PEIRCE is the only author in this country who has used D ; and we have made these remarks for the benefit of those students who meet with this notation only in the MONTHLY. We do not advise the exclusive use of either notation. Use the one most convenient in the particular case. In all cases, however, in which we simply wish to indicate the operation, D is preferable. Thus Dx^n is better than $\frac{d}{dx}x^n$.

2. Find the derivative of $u = a^x$; $\log u = x \log a$. Giving increments $\log(u + du) = (x + dx) \log a$; and taking the ratio,

$$\log(u + du) - \log u = (x + dx) \log a - x \log a;$$

or, $\log\left(1 + \frac{du}{u}\right) = dx \log a.$

Therefore, by the proposition

$$\log\left(1 + \frac{du}{u}\right) = \frac{du}{u} = dx \log a. \therefore \frac{du}{dx} = u \log a = a^x \log a.$$

3. Find the derivative of $u = ax^n$. $\log u = n \log x + \log a$.
 $\log(u + du) = n \log(x + dx) + \log a$. Hence

$$\log(u + du) - \log u = n \log(x + dx) - n \log x;$$

or, $\log\left(1 + \frac{du}{u}\right) = n \log\left(1 + \frac{dx}{x}\right) \therefore \frac{du}{u} = \frac{n dx}{x}; \frac{du}{dx} = n a x^{n-1}.$

The constant factor in the function is still in the derivative.

4. Find the derivative of $u = x^{\frac{p}{q}}$. $\log u = \frac{p}{q} \log x$.

$$\log(u + du) = \frac{p}{q} \log(x + dx); \log\left(1 + \frac{du}{u}\right) = \frac{p}{q} \log\left(1 + \frac{dx}{x}\right).$$

$$\therefore \frac{du}{u} = \frac{p}{q} \cdot \frac{dx}{x}; \text{ or } \frac{du}{dx} = \frac{p}{q} \cdot \frac{u}{x} = \frac{p}{q} x^{\frac{p}{q}-1}.$$

Hence the rule for finding the derivative of a power of a variable is precisely the same for all exponents, whether integral or fractional, and, as is readily seen, whether positive or negative.

5. Find the derivative of $u = x^n + C$, C being a constant quantity.
 $u - C = x^n$; $\log(u - C) = n \log x$. $\log(u - C + du) = n \log(x + dx)$;

$$\log\left(1 + \frac{du}{u - C}\right) = n \log\left(1 + \frac{dx}{x}\right). \therefore \frac{du}{u - C} = \frac{n dx}{x};$$

or $\frac{du}{dx} = \frac{n(u - C)}{x} = n x^{n-1}.$

Therefore, a constant which is not a factor disappears in the derivative.

6. Find the derivatives of $u = \log x$. $u + du = \log(x + dx)$;
 $du = \log(x + dx) - \log x = \log\left(1 + \frac{dx}{x}\right) = \frac{dx}{x} \therefore \frac{du}{dx} = \frac{1}{x}.$

7. *Functions of more than one variable.* Find the derivative of
 $u = x^m y^n$. $\log u = m \log x + n \log y$.

$$\log(u + du) = m \log(x + dx) + n \log(y + dy)$$

$$\log\left(1 + \frac{du}{u}\right) = m \log\left(1 + \frac{dx}{x}\right) + n \log\left(1 + \frac{dy}{y}\right).$$

$$\therefore \frac{du}{u} = \frac{m dx}{x} + \frac{n dy}{y}; \text{ or, } du = m x^{m-1} y^n dx + n y^{n-1} x^m dy.$$

If either x or y be made the independent variable, the derivatives are

$$D_x u = m x^{m-1} y^n + n y^{n-1} x^m D_x y; D_y u = m x^{m-1} y^n D_y x + n y^{n-1} x^m.$$

The values of these derivatives, $D_x u$ and $D_y u$, are not so symmetrical in their form as the value of the differential du ; which is true, in general, of functions of more than one variable. It is therefore, usually better to retain the differential form.

8. Find the differential of the fraction $u = \frac{x}{y}$.

$$\log u = \log x - \log y; \log(u + du) = \log(x + dx) - \log(y + dy).$$

$$\log\left(1 + \frac{du}{u}\right) = \log\left(1 + \frac{dx}{x}\right) - \log\left(1 + \frac{dy}{y}\right). \therefore \frac{du}{u} = \frac{dx}{x} - \frac{dy}{y}.$$

$$\therefore du = \frac{u dx}{x} - \frac{u dy}{y} = \frac{dx}{y} - \frac{x dy}{y^2} = \frac{y dx - x dy}{y^2}.$$

9. Find the differential of $u = y^x$. $\log u = x \log y$.

$$\log(u + du) = (x + dx) \log(y + dy) = x \log(y + dy) + dx \log(y + dy).$$

$$= x \log(y + dy) + dx \log y \left(1 + \frac{dy}{y}\right).$$

$$= x \log(y + dy) + dx \log y + dx \log\left(1 + \frac{dy}{y}\right).$$

$$\log\left(1 + \frac{du}{u}\right) = x \log\left(1 + \frac{dy}{y}\right) + dx \log y + dx \log\left(1 + \frac{dy}{y}\right).$$

$$\therefore \frac{du}{u} = \frac{x dy}{y} + dx \log y + \frac{dx dy}{y}.$$

$$du = \frac{x u dy}{y} + u dx \log y = x y^{x-1} dy + y^x dx \log y.$$

It will be seen, that $\frac{dx dy}{y}$, which is of the second order, is omitted.

NOTES ON THE THEORY OF PROBABILITIES.

By SIMON NEWCOMB, Nautical Almanac Office, Cambridge.

13. SOME theorems relating to combinations will be given here for convenience of reference.

If we have a set of n things, the number of different ways in which a collection of s things can be taken from it, is called the number of combinations of s in n , and is represented by the notation $\overset{s}{C}_n$, or $C(s, n-s)$. Since, for every collection of s things which can be taken, there will remain a collection of $n-s$ things not taken, we may consider the number of combinations of s in n as being the number of ways in which a collection of s things can be divided into two sets, one containing s things, the other $n-s$. It is shown in treatises on Algebra, that

$$(1) \quad \overset{s}{C}_n = \frac{n(n-1)(n-2) \dots (n-s+1)}{1.2.3 \dots s} = \frac{n!}{(n-s)!s!}.$$

From the above, we may easily deduce the following equations :

$$(2) \quad \overset{s+1}{C}_s = \frac{n-s}{s+1} \overset{s}{C}_n; \quad \overset{s-1}{C}_n = \frac{s}{n-s+1} \overset{s}{C}_n; \quad \overset{s}{C}_n = \overset{n-s}{C}_n; \quad \overset{0}{C}_n = \overset{n}{C}_n = 1.$$

Using the notation of combinations, the binominal theorem may be expressed in the form

$$(3) \quad (a+x)^n = \overset{0}{C}_n a^n + \overset{1}{C}_n a^{n-1}x + \overset{2}{C}_n a^{n-2}x^2 + \dots = \sum_{s=0}^n \overset{s}{C}_n a^{n-s} x^s.$$

Suppose that the set is divided into two classes, white and black, for example ; then a collection of s things may be composed of any s white things,

or any $s-1$ white combined with any 1 black,

"	$s-2$	"	"	2	"
"	&c.	"	"	&c.	"
"	1	"	"	$s-1$	"
"	0	"	"	s	"

from which we deduce the general theorem

$$(4) \quad \begin{matrix} s \\ C \end{matrix} = \begin{matrix} s \\ C \end{matrix} + l \begin{matrix} s-1 \\ C \end{matrix} + \begin{matrix} 2 \\ C \end{matrix} \begin{matrix} s-2 \\ C \end{matrix} + \&c. \dots + \begin{matrix} s-1 \\ C \end{matrix} \begin{matrix} 1 \\ C \end{matrix} + \begin{matrix} s \\ C \end{matrix} = \sum_r \begin{matrix} r \\ C \end{matrix} \begin{matrix} s-r \\ C \end{matrix}$$

where l may be any number (at pleasure) less than n .

If we represent the number of ways in which a number of things equal to $(s + s' + s'' + \dots \&c.)$ can be divided into sets, of which one shall contain s things, another s' , &c., by $C(s, s', s'' \dots)$, we have

$$C(s, s', s'' \dots) = \frac{(s + s' + s'' + \dots)!}{s! s'! s''! \dots}$$

14. Illustrative Problems.

I. A bag contains b black balls, and b' white ones. A number s of balls being drawn, what is the probability that they will all be white?

$\begin{matrix} s \\ C \end{matrix}_{b+b'}$ will be the whole number of combinations of s balls in the bag, any one of which may be drawn. Of these combinations, only $\begin{matrix} s \\ C \end{matrix}_{b'}$ will consist entirely of white balls. The probability required is therefore $\begin{matrix} s \\ C \end{matrix}_{b'} \div \begin{matrix} s \\ C \end{matrix}_{b+b'}$. In the same way the probability of drawing s black balls is found to be $\begin{matrix} s \\ C \end{matrix}_b \div \begin{matrix} s \\ C \end{matrix}_{b+b'}$.

EXAMPLE. If there are two white balls and eight black ones, of which two are drawn, there will be forty-five combinations of 2 in 10, any one of which may be drawn. But only one of these combinations will consist entirely of white balls, and the probability of drawing that combination will be $\frac{1}{45}$, and the odds against it 44 to 1. We might arrive at the same result by the principle of § 10. If we suppose the balls to be drawn in succession, the probability that the first ball drawn will prove white is $\frac{2}{10}$, since of 10 balls 2 are white. On the supposition that this event occurs, the probability that the second ball will also be white is $\frac{1}{9}$, since of 9 balls one will

be white. The probability of the compound event is therefore $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

II. To find the probability, that, at whist, a player other than the dealer holds any given number of trumps.

The trump turned up belongs to the dealer, there will remain 51 cards, 12 trumps, and 39 non-trumps to be divided at random, the dealer taking 12, and each of the players 13. In order that a player may have no trump, his 13 cards must come entirely from the 39 non-trumps, and the number of ways in which this result can happen is $\frac{13}{39}$. A hand containing one trump only may be composed of any one of the 12 trumps combined with any combination of 12 in the 39 non-trumps, and such a hand can happen in $12 \times \frac{12}{39}$ different ways. In general, a hand containing s trumps may consist of any s of the 12 trumps combined with any $13 - s$ of the 39 non-trumps, and there will be $\frac{12}{12} \times \frac{13-s}{39}$ such hands. The probability of holding s trumps will therefore be $\frac{12}{12} \times \frac{13-s}{39} \div \frac{13}{51}$.

If, in the above expression, we give to s the successive values 0, 1, 2, &c., we shall find

a probability of .02	that the player has no trump,
" "	.10 " " 1 "
" "	.23 " " 2 trumps,
" "	.30 " " 3 "
" "	.22 " " 4 "
" "	.10 " " 5 "
" "	.03 " " 6 "
a very small probability	" " 7 or more.

By a process of reasoning similar to the above we shall find the

probability that the dealer has s trumps to be

$$\frac{s-1}{12} \times \frac{13-s}{39} \div \frac{12}{51}.$$

III. A set of dice being thrown from a box, to find the probability that any given system of numbers will be thrown.

Suppose for the sake of clearness, that the several dice are distinguished as the first, second, &c. Then any one of the six sides of the first die may be combined with any one of the six sides of the second, making 6^2 on two dice. Any one of these combinations may be combined with any side of the third, making 6^3 combinations on three dice; and by continuing the same process it is seen, that, in general, the number of combinations on n dice is 6^n . The given system of numbers may be thrown in as many ways as there are permutations of the numbers composing it; and if we represent the number of permutations by P , the probability required will be $\frac{P}{6^n}$. This will be easily understood by comparing the following examples:

1. Suppose that there are three dice of different colors, — say white, yellow, red, and let it be required to find the probability that the numbers 1, 2, 3, will be thrown. This system can be thrown on the three dice in the six following ways:

White,	1	1	2	2	3	3
Yellow,	2	3	1	3	1	2
Red,	3	2	3	1	2	1

The probability of each way being $\frac{1}{216}$, the probability required will be $\frac{6}{216}$.

2. If two aces and a deuce were required to be thrown, it could be done only in the following ways:

White,	1	1	2
Yellow,	1	2	1
Red,	2	1	1

The probability of its being thrown is therefore $\frac{1}{4}$.

3. The probability of throwing three aces is $\frac{1}{216}$, since they can be thrown in only one way.

The proposition, that an ace, deuce, and three are six times as likely to be thrown as three aces, might at first sight seem paradoxical. A comparison of the above will, however, make it quite clear.

NOTE ON MAXIMA AND MINIMA.

BY LEWIS R. GIBBES,
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THE method used in the Note on the cycloid may be applied to other problems involving consideration of maxima or minima, and with advantage in such branches as are studied without supposing in the student a knowledge of the Calculus, or in questions which may be presented to him before he has yet entered upon the higher parts of his intended mathematical course.

We will apply it to an old problem, often mentioned in works on descriptive astronomy, but seldom solved, the result only being given, the student being supposed unacquainted with the Calculus. Observation shows that the planet Venus passing from inferior conjunction towards greatest elongation, increases in brilliancy, attains a maximum brightness at a certain point, and then declines as it advances towards superior conjunction. Hence the following

PROBLEM. *To find the position of a planet at the time of its greatest brilliancy, as seen from another whose orbit includes its own.*

We will suppose the orbits of both to be circular and in the same plane, which supposition will be found not to affect the generality of the conclusion.

Let r be the radius of the orbit of the interior planet, and r' the radius of the exterior; also, let z be the distance between the two planets at any time, and B the brightness at the time, the unit of brilliancy being the brightness of the unit of area of the disc of the planet, at the unit of distance from the Sun and from the exterior planet. By trigonometry, the cosine of the angle at the exterior planet is

$$(1) \quad \cos \frac{r'}{z} = \frac{z^2 + r'^2 - r^2}{2 z r'}.$$

The cosine of angle at interior planet is

$$(2) \quad \cos \left(180^\circ - \frac{r}{z}\right) = \frac{z^2 + r^2 - r'^2}{2 z r}.$$

From optics, we have

$$B = \frac{\text{illuminated area of disc}}{z^2 r^2} = \frac{\text{a semicircle} \pm \text{a semi-ellipse}}{z^2 r^2},$$

the semi-major axis of the ellipse is equal to the radius of the semicircle, and the semi-minor axis is equal to that radius multiplied by $\cos \left(180^\circ - \frac{r}{z}\right)$, and when the semi-ellipse is subtractive, the cosine is negative, so that the illuminated area is always proportional to $1 + \cos \left(180^\circ - \frac{r}{z}\right)$, and we may put $B = \frac{1 + \cos \left(180^\circ - \frac{r}{z}\right)}{z^2 r^2}$ or by (2) $= \frac{z^2 + 2 z r + r^2 - r'^2}{2 z^3 r^2}$. Hence we have

$$(3) \quad 2 r^3 B = \frac{1}{z} + 2 r \frac{1}{z^2} + (r^2 - r'^2) \frac{1}{z^3}.$$

Let z_0 be the distance of the interior planet from the exterior at the point of maximum brilliancy; then there can be found pairs of points, one member of each pair on each side of that point, at which the brilliancy, though less than the maximum, must be equal. Let z_1 and z_2 be the distances of the interior planet from the exterior one at the two points of one of these pairs, and let B_1 be the brightness at those points, the same for each. Then we shall have these two equations similar in form to (3):

For the first point, $2r^3 B_1 = \frac{1}{z_1} + 2r \frac{1}{z_1^2} + (r^2 - r'^2) \frac{1}{z_1^3}$.

For the second point, $2r^3 B_1 = \frac{1}{z_2} + 2r \frac{1}{z_2^2} + (r^2 - r'^2) \frac{1}{z_2^3}$.

By subtraction, $0 = \frac{1}{z_1} - \frac{1}{z_2} + 2r \left(\frac{1}{z_1^2} - \frac{1}{z_2^2} \right) + (r^2 - r'^2) \left(\frac{1}{z_1^3} - \frac{1}{z_2^3} \right)$.

Multiplying by $\frac{z_1^2 z_2^2}{z_2 - z_1}$, $0 = z_1 z_2 + 2r(z_1 + z_2) + (r^2 - r'^2) \frac{z_1^2 + z_1 z_2 + z_2^2}{z_1 z_2}$.

As this equation holds good for every such pair of points, however near to the point of maximum brilliancy, it holds good when they coincide with it; but then $z_1 = z_2 = z_0$, and $z_1^2 = z_2^2 = z_1 z_2 = z_0^2$; hence, substituting in the above equation z_0 and z_0^2 for the quantities to which they are then equal, we have the equation of the maximum

$$0 = z_0^2 + 4r z_0 + 3(r^2 - r'^2).$$

Whence we get

$$(4) \quad z_0 = (r^2 + 3r'^2)^{\frac{1}{2}} - 2r, \text{ or } \frac{z_0}{r'} = \left(\frac{r^2}{r'^2} + 3 \right)^{\frac{1}{2}} - \frac{2r}{r'},$$

$$(5) \quad r = \left(\frac{1}{3} z_0^2 + r'^2 \right)^{\frac{1}{2}} - \frac{2}{3} z_0, \text{ or } \frac{r}{r'} = \left(\frac{1}{3} \frac{z_0^2}{r'^2} + 1 \right)^{\frac{1}{2}} - \frac{2}{3} \frac{z_0}{r'},$$

rejecting the negative values as not conforming to the conditions of the problem. By substituting this value of r for r in (1), and of z_0 for z in (2), we obtain

$$(6) \quad \cos \frac{r'}{z_0} = \frac{2}{3} \left(\frac{1}{3} \frac{z_0^2}{r'^2} + 1 \right)^{\frac{1}{2}} + \frac{2}{3} \frac{z_0}{r'},$$

$$\cos \left(180^\circ - \frac{r}{z_0} \right) = \frac{3 \frac{r}{r'} + \frac{r'}{r} - 2 \left(\frac{r^2}{r'^2} + 3 \right)^{\frac{1}{2}}}{\left(\frac{r^2}{r'^2} + 3 \right)^{\frac{1}{2}} - 2 \frac{r}{r'}}.$$

With any given value of r less than r' , the distance at greatest brilliancy, z_0 , can be obtained by (4), and the elongation at that time r' , and the annual parallax $\left(180^\circ - \frac{r}{z_0} \right)$ from equations (6).

If $(180^\circ - \frac{r}{z_0}) = 90^\circ$, which can only happen at the greatest elongation, then $\cos(180^\circ - \frac{r}{z_0}) = 0$; hence from second of equations (6), $3\frac{r}{r'} + \frac{r'}{r} - 2(\frac{r^2}{r'^2} + 3)^{\frac{1}{2}} = 0$, and we get $\frac{r^2}{r'^2} = \frac{1}{5} = (0.447)^2$; $\frac{r}{r'}$ is also the sine of the greatest elongation in the orbit whose radius is r , so that when $\frac{r}{r'} = 0.447$ the maximum brilliancy occurs exactly at the greatest elongation, and the elongation is then $26^\circ.33'.53''$.

Since, by the conditions of the problem, $\frac{r}{r'}$ is always less than unity, the denominator of the right hand side of the last of equations (6) is always positive, and the sine of that side will depend on that of the numerator; this will be negative when $\frac{r^2}{r'^2} > \frac{1}{5}$ or $\frac{r}{r'} > 0.447$, and positive in the contrary case, and the negative cosine gives $(180^\circ - \frac{r}{z_0}) > 90^\circ$, or maximum brilliancy occurs between greatest elongation and inferior conjunction; the positive cosine shows that it occurs between greatest elongation and superior conjunction.

If $\frac{r}{r'} = \frac{1}{4}$, then $\cos(180^\circ - \frac{r}{z_0}) = +1$; or $(180^\circ - \frac{r}{z_0}) = 0^\circ$; that is, if $\frac{r}{r'} = \frac{1}{4}$ or less, the maximum brilliancy occurs only at superior conjunction.

If $z_0 = 0$, then from first of equations (6), $\cos \frac{r}{z_0} = \frac{2}{3}$; that is, however near the interior planet may approach the exterior, the elongation at which the greatest brilliancy will occur, cannot exceed $48^\circ.11'.22''$. Hence if $\frac{r}{r'}$ be

$\left\{ \begin{array}{l} < 1 \text{ and } > 0.447 \\ = 0.447 \\ < 0.447 \text{ and } > 0.250 \\ = \text{ or } > 0.250 \end{array} \right.$	$\left\{ \begin{array}{l} \text{max.} \\ \text{brill.} \\ \text{will} \\ \text{occur} \end{array} \right.$	$\left\{ \begin{array}{l} \text{between inf. conj. and gr. elong.} \\ \text{at greatest elongation,} \\ \text{between gr. elong. and sup. conj.} \\ \text{at superior conjunction,} \end{array} \right.$	$\left\{ \begin{array}{l} \text{elong.} \\ \text{will} \\ \text{then} \\ \text{be} \end{array} \right.$	$\left\{ \begin{array}{l} \text{betw. } 48^\circ.11' \text{ and } 26^\circ.34' \\ 27^\circ.34' \\ \text{between } 26^\circ.34' \text{ and } 0^\circ \\ 0^\circ \end{array} \right.$

From this table, it will be seen that the statement made in some works (see OLMSTED'S Astronomy, article 311 of successive editions) is erroneous, that "an inferior planet is brightest at a certain point between its greatest elongation and inferior conjunction." The mean radius vector of Mercury being 0.387, he in general arrives at greatest brightness between greatest elongation and superior conjunction.

If the exterior planet be supposed fixed at a certain point in its orbit, the points, at each of which occurs the maximum brilliancy for successive values of r , will lie in a curve whose equation we proceed to find. Let the fixed position of the exterior planet be the origin of rectangular coördinates, the corresponding radius vector being the axis of x positive towards the sun; and let x and y be the coördinates of any point in the curve whose distance from the superior planet is z , and from the sun is r , continuing to express by r' the distance of the exterior planet from the sun. Then we have by geometry

$$\left. \begin{aligned} z^2 &= y^2 + x^2 \\ (7) \quad \text{also} \quad r^2 &= y^2 + (r' - x)^2 \\ \text{and from equation (4),} \quad z^2 &= 5r^2 - 4(r^4 + 3r'^2 r^2)^{\frac{1}{2}} + 3r'^2 \end{aligned} \right\}$$

and by combining these equations so as to eliminate z and r , we will obtain the equation of the curve. This elimination will most easily be effected thus: Subtract the first equation from the sum of the second and third, simplify, clear of radicals and reduce, and there will result $(r' + x)r^2 = r'^2(r' - x) + \frac{1}{4}r'x^2$; now, by substituting the value of r^2 from the second of the above equations (7) and reducing, we get, finally, $y = \pm x \left(\frac{\frac{5}{4}r' - x^{\frac{1}{2}}}{r' + x} \right)$. This equation shows the curve to be a defective hyperbola of NEWTON'S 41st species of lines of the third order.

Since to every value of x there are two equal values of y with contrary sines, the axis of abscissas is the axis of the curve, and,

since there is but one constant r' in the equation, all such curves are similar. If x be positive and greater than $\frac{5}{4}r'$, y is impossible, no part of the curve being at a greater distance beyond the Sun from the exterior planet than $\frac{1}{4}$ of the radius of its orbit; if x be positive and equal to $\frac{5}{4}r'$, then $y = 0$, the curve cuts the axis of x at a distance beyond the Sun equal to $\frac{1}{4}r'$, which corresponds to the case already mentioned of the maximum brilliancy at superior conjunction; the value of y for $x = +r'$ is equal to that for $x = +\frac{1}{2}r'$, each equal to $r' \times \frac{1}{4}\sqrt{2}$, and this is also the value of y corresponding to $x = -\frac{1}{4}r'$; the three values of y for the three values of x , $+\frac{7}{8}r'$, $+\frac{1}{4}r'$, $-\frac{5}{8}r'$ are to each other in the ratio of 7, 4, 25; the value of y for $x = -\frac{3}{4}r'$ is to that for $x = -\frac{1}{4}r'$ as 6 to 1; if $x = \frac{1}{8}r'$, then $y = x$; if $x = 0$, then $y = 0$; the origin of coördinates is a multiple point, two branches of the curve passing through it; lastly, if $x = -r'$, then y is infinite, the Hyperbola has an asymptote, cutting the axis at right angles at a point distant from the Sun twice the radius of the orbit of the exterior planet. The curve comes in from infinity on one side of the axis, along the asymptote, bending gradually towards the point at which the exterior planet is supposed fixed, passes to the other side of the axis through that point, curves in a loop round the Sun, cutting the axis beyond the Sun at a distance already indicated, completes the loops by passing a second time through the fixed position of the exterior planet, and then, bending away from the axis, recedes to infinity, approaching the asymptote.

If the polar equation to the curve be desired, it can easily be had from the first of equations (6), the pole being at the same point as the origin of rectangular coördinates, $\frac{z_0}{r'}$ being the polar angle, reckoned from the axis of abscissas, and $\frac{z_0}{r'}$ being the radius vector, taking r' as unity; we thus get $\frac{z_0}{r'} = \frac{3}{4} \cos \frac{z_0}{r'} - \sec \frac{z_0}{r'}$.

The portion of the curve belonging to the negative values of x fulfil the algebraical and also geometrical conditions of the problem, but not the optical conditions; presenting another example of a class of cases, so puzzling to some persons, in which an algebraical expression, full of significance when applied to one question, is totally devoid of meaning when applied to another in which the fundamental conceptions are quite different. A planet in a circular orbit, seen from an interior one, will be brightest when in opposition.

So far the interior planet has been supposed to move in a circular orbit; if it move in an elliptic or other orbit, it will evidently be brightest at that point in its orbit which cuts the curve of maximum brilliancy belonging to the corresponding position of the superior planet, at the same moment. If there should be two or more such intersections at different points, there will be two or more maxima of brightness. The intensity of brightness will not be the same for both maxima; for if a planet were to move in the curve of maximum brilliancy from superior to inferior conjunction, the intensity of brightness at successive points, though a maximum for the given distance from the Sun at any one point, would diminish from superior conjunction to a point about equally distant from the Sun and from the exterior planet, at which would occur a *minimum maximorum*, and then increase again to inferior conjunction. At the *minimum maximorum*, where z_0 and r would each be about $\frac{2}{3}$ of r' , the brightness would be little less than one fourth its intensity at the points where $r = \frac{1}{4} r'$ and $r = \frac{3}{16} r'$, at which two points the brilliancy would be nearly equal.

If the method above used be applied to determine the value of x for which y is a maximum, we shall find $x = + r' \times 0.763$, and the corresponding value of $y = r' \times 0.401$. An ellipse described about the Sun as a focus, having the radius vector to the fixed position of the

exterior planet as a line of apsides, the aphelion between the Sun and the superior planet, a perihelion distance equal to $0.250 r'$, and a semi-minor axis equal to $0.401 r'$, will closely approximate to the curve of maximum brightness throughout its perihelion half; so that a planet would always be near this curve through half its orbit, if that orbit were the above ellipse, whose mean distance would be $0.445 r'$, and eccentricity $0.195 r'$. The aphelion half would lie within the loop of the curve. The ellipse which would most closely approximate to the loop throughout its greatest extent would have its perihelion at the same point as that of the preceding one, a major axis of about $1.150 r'$, and a minor axis of about $0.800 r'$.

The above problem is as old as the time of HALLEY, who first proposed and solved it for the planet Venus, and his result is that generally quoted in the elementary works; he also remarked the limit which occurs when $r = \frac{1}{4} r'$. The other results given above we have not met with anywhere; but it is not impossible that, in the *Berlin Memoirs* or other equally inaccessible treasuries of science, all our results have been anticipated years ago, if not a century since. Even if so antiquated, they may be new to many of our readers.

ARCS OF GREAT AND SMALL CIRCLES.

BY GEORGE P. BOND,
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IF a plane intersects the surface of a sphere without passing through its centre, a small circle, c , is formed by the intersection. The great circle, C , most nearly representing it through its entire circumference, will have its plane parallel to that of c . If it is only required to represent in the best manner an arc of c less than a

circumference, by means of a corresponding arc of C , the planes of the two circles must be inclined by an angle depending on the length of the arc, and the distance of the plane of c from the centre of the sphere. The direction of the line of intersection of the planes will be perpendicular to a diameter passing through the middle point of the arc.

In the case where c differs but little from a great circle, let ψ be the angle between the planes of C and c ; if G be the great circle to which c is secondary, ψ will be the angle between C and G . Let also ζ be the angle measured from their intersection to a point on C . The distance of c from C at this point measured on the surface of the sphere will be

$$(1) \quad x = \psi \sin \zeta - \gamma,$$

γ being the distance of c from G .

In order to adjust an arc of C comprised between the limits ζ_1 and ζ_0 , so that the mean of the squares of the deviations of each infinitesimal element from its corresponding element of an arc of c shall be the least possible, the position at C must satisfy the conditions

$$(2) \quad \psi = 4 \frac{\cos \zeta_0 - \cos \zeta_1}{(2 \zeta_1 - \sin 2 \zeta_1) - (2 \zeta_0 - \sin 2 \zeta_0)} \gamma, \quad \frac{\zeta_1 + \zeta_0}{2} = 90^\circ.$$

The general expression for the mean value of x^2 is

$$(3) \quad \frac{\psi^2}{2} - \frac{\psi^2}{4} \left(\frac{\sin 2 \zeta_1 - \sin 2 \zeta_0}{\zeta_1 - \zeta_0} \right) + 2 \psi \gamma \frac{\cos \zeta_1 - \cos \zeta_0}{\zeta_1 - \zeta_0} + \gamma^2,$$

which becomes, when the adjustment is made in conformity with the method of least squares

$$(4) \quad \gamma^2 \left(1 - \frac{2 \eta^2}{1 + \eta \cos \theta} \right),$$

where

$$\theta = \frac{\zeta_1 - \zeta_0}{2}, \quad \eta = \frac{\sin \theta}{\theta}.$$

If we take the square root of the mean of the square of the

deviations of each of its elements for the mean error, irrespective of signs, of the arc of C considered as representing a corresponding arc of c , we have from (4) the following results:

	Included arc, $\zeta_1 - \zeta_0 = 0^\circ$	Mean error = $\pm 0.0000 \gamma$,
	“ “ = 30	“ = 0.0100γ ,
(5)	“ “ = 60	“ = 0.0421γ ,
	“ “ = 90	“ = $\pm 0.0975 \gamma$.

It is evident that where the included arc as well as γ is small, the approximation reaches a high degree of accuracy, the error varying nearly as the product of γ into the square of the arc.

The above conclusions may be applied to the theory of the transit instrument. The path described by the optical axis of the transit is the small circle, c , distant from the great circle, G , to which it is secondary by the small arc γ . G is commonly referred to the meridian by the small arcs α and β , which are its distances from the meridian at the horizon and at the zenith respectively. If φ is the latitude of the place of observation, and δ the declination of the object towards which the instrument is directed, the deviation of the small circle c from the meridian will be

$$(6) \quad \alpha \sin(\varphi - \delta) + \beta \cos(\varphi - \delta) + \gamma;$$

which may be reduced to the form

$$(7) \quad P \sin(Q - \delta),$$

P and Q being constant for arcs for which the errors of deviation of c from C are insensible. With the usual adjustment of the instrument we may infer from (5) that its path for arcs of thirty or forty degrees may be treated as a great circle without appreciable error.

The same general conclusion may be reached in another way. If α and β are eliminated from the equations

$$(8) \quad \begin{aligned} \alpha \sin(\varphi - \delta_1) + \beta \cos(\varphi - \delta_1) + \gamma &= \mu_1, \\ \alpha \sin(\varphi - \delta_2) + \beta \cos(\varphi - \delta_2) + \gamma &= \mu_2, \\ \alpha \sin(\varphi - \delta_3) + \beta \cos(\varphi - \delta_3) + \gamma &= \mu_3, \end{aligned}$$

we have

$$(9) \quad 4 \sin \frac{\delta_1 - \delta_2}{2} \sin \frac{\delta_1 - \delta_3}{2} \sin \frac{\delta_2 - \delta_3}{2} \gamma = \mu'''.$$

The coefficient of γ is a small quantity of the third order when the included arcs are of the first order, while the second member of (9) will be affected by errors of observation multiplied by small coefficients of the first order. When γ is small, of the order of errors of observation, it is plain that for arcs of moderate extent, both (8) and (9) may be satisfied by the value

$$\gamma = 0,$$

which reduces the path described, within the limits assigned, to an arc of a great circle.

ON MR. COLLINS'S PROPERTY OF CIRCULATES.*

BY JAMES EDWARD OLIVER, Lynn, Mass.

LET $C\xi$ denote a circulate of ξ places; that is, a number formed, except perhaps its left-hand portion, by the endless repetition of ξ figures in the same order. Let $F()$ and $M()$ be the greatest common factor and the least common multiple of the inclosed numbers. Let l_1, l_2 , &c. be subfactors or multiples of certain of the A numbers x, y, z , or of previous values of l_i ; and form λ_1, λ_2 , &c. in a precisely corresponding manner from ξ, η, ζ . Let the arbitrary integers, w, w_1 , &c. be exact divisors of any numerators written over them; — and call a the base of the system of numeration employed.

* See page 295.

From the periods of several circulates, to find the period of their product. The A conditions

$$(1) \quad \frac{x'}{x} = C\xi, \quad \frac{y'}{y} = C\eta, \quad \dots \frac{z'}{z} = C\zeta,$$

the fractions and the periods being in their simplest terms, are equivalent to

$$(2) \quad a^\xi \equiv 1 [x],^* \quad a^\eta \equiv 1 [y], \quad \dots a^\zeta \equiv 1 [z];$$

where, as in all that follows, the exponents are assumed to be *minimum* positive roots of their congruences. Whence

$$(3) \quad a^{F(\xi, \eta, \zeta)} \equiv 1 [F(x, y, z)],$$

because all the roots of $a^q \equiv 1 [F(x, y, z)]$ (3'), such as ξ, η, ζ , are multipliers of q_0 , the least positive root; hence their greatest common measure $F(\xi, \eta, \zeta)$ is a multiple of q_0 ; hence $F(\xi, \eta, \zeta)$ is a root of (3').

Again, since modulus x requires that the exponent of a be a multiple of ξ , while modulus y requires the exponent to contain η , and so on, we have

$$(4) \quad a^{M(\xi, \eta, \zeta)} \equiv 1 [x, y, z, \therefore M(x, y, z)].$$

From (3) and (4),

$$(5) \quad a^{l_s} \equiv 1 [l_s]. \quad (6) \quad \therefore l_s = \frac{a^{l_s} - 1}{w_s}.$$

$$\begin{aligned} \text{From (4),} \quad & \left(a^{M(\xi, \eta, \zeta)} = 1 + w' \cdot M(x, y, z) \right)^{\frac{x \cdot y \cdot z}{M(x, y, z)}} \\ & = 1 + w' \cdot x \cdot y \cdot z \left(1 + \frac{(x \cdot y \cdot z - M(x, y, z))}{1 \cdot 2} w' + \&c. \right). \end{aligned}$$

But $\frac{(x \cdot y \cdot z - M(x, y, z)) \cdot (x \cdot y \cdot z - s \cdot M(x, y, z))}{1 \cdot 2 \cdot \dots \cdot (s-1)}$ is an integer; for since

* See note at end of this article.

$M(x, z)$ divides $(x..z)$, one of every k consecutive terms in the numerator is divisible by k . Hence

$$(7) \quad a^{M(\xi, \zeta) \frac{x..z}{M(x, z)}} \equiv 1 + w'' \cdot x..z \quad \therefore \equiv 1 [x..z].$$

$$(8) \quad \therefore C\xi \cdot C\eta \dots C\zeta = \frac{x'..z'}{x..z} = C \cdot \frac{x..z}{M(x, z)} M(\xi, \zeta).$$

It remains to select such forms of l_1, l_2 , that $\frac{x..z}{M(x, z)}$ may be some simple function ψ of them; then by substituting for l_1, l_2 , their values (6), the period of $C\xi \dots C\zeta$ is expressed in terms of λ_1, λ_2 ,; that is, of ξ, ζ .

Consider the $2^A - 1$ factors, 2^{A-1} of which enter each letter x, z ; one entering only y , &c., and one entering all x, z . Usually most of these factors are unity. Call P_s the product of the $\frac{A \cdot (A-s+1)}{1 \dots s}$ factors that enter s letters each. $M(x, z) = P_1 P_2 \dots P_A$, $x..z = P_1 P_2^2 \dots P_A^A$, $\therefore \psi = P_2 P_3^2 \dots P_A^{A-1}$, and requires at least $A - 1$ functions l_s , even if any l_s be squared, cubed, &c., since no l_s is above the first degree in P_A or P_s .

We may take l_s = the least common multiple of the $\frac{A \cdot (A-s+1)}{1 \dots s}$ greatest common factors of x, z taken s at a time, $\therefore l_s = P_s P_{s+1} \dots P_A$. Or l_s may variously be taken unsymmetrical as to $(x_1, x_A = x, z)$; for instance $l_2 = F(x_2, x_1)$, $l_3 = F(x_3, M_{x_2, x_1})$, $l_4 = F(x_4, M_{x_3, x_1})$ &c.; or &c.; all which may be further varied by permutating any of the letters x, z that enter unsymmetrically. In these cases, $\psi = l_2 \dots l_A$, and (8) becomes

$$(9) \quad C\xi \dots C\zeta = C \cdot \frac{(a^{\lambda_2} - 1) \dots (a^{\lambda_A} - 1)}{w} M(\xi, \zeta),$$

where $w = w_2 \dots w_A$, and whose constants λ_2, λ_A are found from ξ, ζ as l_2, l_A would be from x, z . In the symmetrical case, l_s is the product of those highest powers of the primes α, β, γ , which occurs

each in some s of ξ, ζ ; so that each λ , \therefore also each $a^\lambda - 1$, divides all preceding ones.

EXAMPLE. If $\kappa_1, \dots, \kappa_A$ be all prime to each other,

$$(10) \quad C\kappa_1 \varrho \dots C\kappa_A \varrho = C \cdot \frac{(a^p - 1)^{A-1}}{w} \kappa_1 \dots \kappa_A \varrho.$$

λ_s may exceed $A - 1$ in number, or the form of ψ may vary, or both; as,

$$(11) \quad C\xi. C\eta. C\zeta = C \left(\frac{w'}{Q^{p_3} - 1} \cdot \frac{(a^{\lambda_\xi} - 1)(a^{\lambda_\eta} - 1)(a^{\lambda_\zeta} - 1)}{w} M(\xi, \eta, \zeta) \right),$$

where $\lambda_\xi = M(F_{\xi, \eta}, F_{\xi, \zeta}) = F(M_{\xi, \eta}, M_{\xi, \zeta})$, λ_η and $\lambda_\zeta = \&c.$

(9 — 11) &c. give all the *simple* periods of the required product, but may also give other, *merely multiple* periods, a criterion for whose suppression would be desirable. For (3) and (4), though *necessary*, are not *sufficient* to (2). Hence (9), (11), &c., or even the different *forms* of either that come from different values of λ_s may not be identical; and any factor not common to all, should be suppressed by conditioning the divisors and multipliers $w_s, w^{(s)}$.

Mr. COLLINS'S case of (10), that

$$C\kappa \varrho. C\tau \varrho = C \cdot \frac{a^p - 1}{w} \kappa \tau \varrho,$$

κ being prime to τ , gives by inversion the period of the quotient of two circulates. If $C\delta = C\pi \div C\kappa$, divide κ into factors κ, ϱ , let $\delta = \tau \varrho$, and have τ prime to κ .

$$C\kappa \varrho. C\tau \varrho = C\pi. \quad \therefore \frac{a^p - 1}{w} \kappa \tau \varrho = \pi.$$

$$(12) \quad \therefore \delta = \frac{w}{a^p - 1} \frac{\tau \varrho}{\kappa}, \quad \left(\frac{\delta}{\varrho} \text{ prime to } \frac{\kappa}{\varrho} \right),$$

where ϱ is successively all factors of κ , but $w < a^p$ may need further restriction. By similar steps, if $C\delta = \frac{C\pi}{C\kappa_1 \dots C\kappa_A}$, we divide $\kappa_1, \dots, \kappa_A$

and θ into factors $\alpha_1 \varrho, \alpha_A \varrho$ and $\tau \varrho$, having τ prime to α_1, α_A . Let $\lambda_2 = F(\lambda_2, \lambda_1)$, $\lambda_3 = F(\lambda_3, M(\lambda_2, \lambda_1))$, &c. Since $\lambda_{A+1} = F(\theta, M(\lambda_A, \lambda_1))$ reduces to ϱ , and $M(\lambda_1, \lambda_A, \theta)$ to $\frac{\theta}{\varrho} M(\lambda_1, \lambda_A)$, — (9) finally gives

$$(13) \quad \theta = \frac{w}{(a^{\lambda_2} - 1) \dots (a^{\lambda_A} - 1)(a^p - 1)} \cdot \frac{\pi \cdot \varrho}{M(\lambda_1, \lambda_A)} \left(\frac{\theta}{\varrho} \text{ prime to } \frac{\lambda_1}{\varrho}, \frac{\lambda_A}{\varrho} \right).$$

NOTE. — GAUSS'S notation, $x \equiv y \pmod{z}$, which we shall write $x \equiv y [z]$, is read " x is congruous to y with regard to the modulus z ," or " x leaves the same remainder as y does, when divided by z ." The modulus may be omitted when there is no fear of mistaking it. $x \equiv y [z', z'', z''', \dots u]$ may be read, " x leaves the same remainder as y , whether divided by z' , by z'' , or by z''' ; hence also when divided by u ."

The second member of $a^{\zeta} \equiv y$ runs through a regular cycle as ζ increases; so that $a^{\zeta} \equiv b$ has an infinity of roots of the form $\zeta = \zeta_0 + w \cdot \zeta'$, ζ_0 being the least positive root, and ζ' that of $a^{\zeta} \equiv 1$.

If $\frac{p}{q}$ be a circulate of χ places, the division of p (or \therefore of 1.000 &c.) by q gives by definition a remainder of p (or \therefore of 1) after every χ quotient figures; hence, $a^{\chi} \equiv 1 [q]$.

See GAUSS'S *Disquisitiones Arithmeticae*, or SERRET'S excellent *Algebre Superieure*. (Paris, 1843), &c.

SOLUTIONS OF PROBLEMS IN PROBABILITIES.

By SIMON NEWCOMB, Nautical Almanac Office, Cambridge.

1. " A HAS the reputation of telling the truth as often as three times in four, B as often as four times in five, and C as often as four times in seven. When A and B agree in affirming what C denies, what is the probability that A and B tell the truth?" See p. 235.

SOLUTION. The *a priori* probability that A and B tell the truth and C falsifies is $\frac{3}{4} \times \frac{4}{5} \times \frac{3}{7} = \frac{6}{35}$. The *a priori* probability that A and B falsify and C tells the truth is $\frac{1}{4} \times \frac{1}{5} \times \frac{4}{7} = \frac{1}{35}$. The proba-

bility of the proposition asserted by A and B is therefore $\frac{6}{35} \div (\frac{6}{35} + \frac{1}{25}) = \frac{24}{49}$.

2. "A person goes on throwing a common dice until he throws an ace; at whatever throw this occurs (the n th), he is to receive the n th of a dollar. What is the value of his expectation?" See p. 235.

SOLUTION. The probability that an ace will be thrown on any given trial is $\frac{1}{6}$, and the probability against it is $\frac{5}{6}$. The probability, then, that an ace will be thrown on the n th throw and not before is $\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \dots = \frac{1}{6} (\frac{5}{6})^{n-1}$; the probability being compounded of the one favorable and the $n-1$ unfavorable cases. Since the thrower gets $\frac{1}{n}$ th of a dollar on the occurrence of this event, the value of his expectation on the n th throw is $\frac{1}{n} \times \frac{1}{6} (\frac{5}{6})^{n-1}$. Giving n successively all integer values from 1 to ∞ , the complete value of his expectation is

$$\frac{1}{6} (1 + \frac{1}{2} \cdot \frac{5}{6} + \frac{1}{3} \cdot (\frac{5}{6})^2 + \frac{1}{4} (\frac{5}{6})^3 + \frac{1}{5} (\frac{5}{6})^4 + \&c.) = \frac{1}{5} \text{ nep. log } 6.$$

The value of his expectation is therefore 63 cents, nearly.

Mathematical Monthly Notices.

Astronomical Notices. Nos. 1-6. Edited by Dr. F. BRÜNNOW, Ann Arbor, Michigan.

This Periodical is issued in numbers of eight pages each, 8vo, and twenty-four numbers will form one volume; but, when desirable, it will be issued more frequently in semi-numbers, four pages at a time. The Editor says: "The main object I have in view in publishing this new periodical is to secure the regular publication of observations made at the Observatory at Ann Arbor, and of the scientific investigations of the officers of the Observatory in general. It is also my intention to make these NOTICES as useful as possible to practical astronomers, by furnishing them always in the shortest time, as well with reliable ephemerides of newly discovered comets and asteroids for the whole time during which they remain visible, as with proper comparison stars observed here in advance, whenever it is possible."

There could be no doubt of the value of such a periodical in the hands of Dr. Brunnnow, even if the evidence were not before us in the six numbers already issued; and it cannot fail to exert

a beneficial influence upon the students of the University. For every new discovery, every reliable observation, every new orbit, in fact, for every result worthy of permanent record, they have at once the means of publication; and it seems to us that this periodical must act as an additional incentive to draw students of astronomy to this flourishing University.

The objections, as it seems to us, which can be urged against its establishment, are the inconvenience of being under the necessity of consulting so many different journals, and the fact, that it is with difficulty that one devoted especially to astronomy can be sustained in this country.

Biographical and Literary Dictionary of the History of the Exact Sciences; containing References to the Relations and Developments in Mathematics, Astronomy, Physics, Chemistry, Mineralogy, Geology, &c., in all Times and Countries; compiled by J. D. POGGENDORFF, Fellow of the Academy of Sciences of Berlin.

This very important work is to be published in four parts, making a volume of from 1,000 to 1,200 large double-column pages. The first and second parts, of 288 pages each, containing about four thousand names, exclusive of references, and ending with the article "HUDDART," we have examined with sufficient care to be able heartily to recommend it to all in this country devoted to science, or interested in its progress. The spirit of the work may be most clearly and briefly intimated in the language of the compiler.

"The leading principle for this dictionary has been to include all persons connected with the mathematical and inductive sciences, as far as any certain notices of their lives could be gained: a condition securing to the work its biographical-literary character, without permitting it to degenerate into a mere list of names and books. Moreover, it was not the intention of the author to give strict biographies and complete literary indices; such would have inconveniently increased the bulk of the work beyond the power of a single compiler, and disadvantageously restricted its circulation and usefulness; he purposed rather to present, in short sketches, a summary, such as has not yet appeared,—a manual, which may be in the possession of every friend of the inductive sciences, satisfying him on the chief points of date, life, and works of persons active in the field, and providing him at the same time with references to the sources whence more detailed information may be obtained.

"For the last ten years the author has been continually employed in the compilation of this work, in which task he has been particularly aided by the extensive literary-historical treasures of the royal library of Berlin, as well as by the services of several friends who have kindly supplied him with numerous authentic communications from scientific men of the present day. Taking into view the copious material thus collected, it may be confidently asserted that this work will be inferior to no similar one on any other branch of science; and representing as it does a whole library of biographical resources, it will doubtlessly animate to historical study in this sphere, and tend to render the same fruitful."

Ueber die Verbesserung der Planeten-Elemente aus beobachteten Oppositionen, angewandt auf eine neue Bestimmung der Pallas-Bahn. Von Dr. J. G. GALLE, ordentlichen Professor der Astronomie an der Universität zu Breslau.

In the Memoir before us, we have a new determination of the orbit of Pallas. The corrections of the osculating elements for 1810 are based upon twelve oppositions; those best observed between 1816 and 1855. The new orbit now agrees as well with observation as could be expected, considering the length of time elapsed, and the fact that the perturbations by Saturn and Mars have not been taken into account.

The author has used the method given by GAUSS (*Theoria Motus*, Art. 76) for forming the

equations of condition; and although it is not the shortest, it is one which the student can readily comprehend. A short historical sketch, which precedes the investigation, informs us that GAUSS was the first to subject the orbit of Pallas to a systematic investigation, as he did those of the three other asteroids discovered before 1810; that ENCKE next took it, and has now put it in charge of Dr. GALLE, whose investigations have already shown that it could not have fallen into abler hands.

It is to be hoped, that tables of this interesting planet may soon be prepared; which has already been done for Flora and Victoria, by Dr. BRUNNOW; for Egeria, by Prof. PEIRCE; for Astræa and Hygea, by Prof. ZECH; and for Metis and Lutetia, by Mr. OTTO LESSER.

Our readers will recall Dr. GALLE as the astronomer to whom LEVERRIER first communicated his predicted place of Neptune, by means of which the planet was at once identified.

Editorial Items.

THE following gentlemen have sent us solutions of the Prize Problems in the April Number of the MONTHLY.

DAVID TROWBRIDGE, Perry City, Schuyler Co., N. Y., answered all the questions.

JOHN N. BENTON, Earleville, N. Y., answered all the questions.

WILLIAM C. HENCK, Student in Dedham High School, answered I. and II. (CARLOS SLAFTER, Principal.)

C. HERSCHEL, Student in the Lawrence Scientific School, answered all but V.

W. M. STIRLING, Baltimore, Md., answered all but III. and V.

E. W. NEWTON, Student in Marietta College, Ohio, answered all but III. and V. (E. W. EVANS, Professor.)

B. F. CLARKE, Student, Waltham, Mass., answered question I. (Rev. T. HILL, Teacher.)

WM. EGERTON, Student in Baltimore College, answered all the questions. (RICHARD COTTER, Professor.)

O. B. WHEELER, Student in the University of Michigan, Ann Arbor, answered all the questions. (D. WOOD, Professor.)

CHARLES BETTLE, Sophomore Class, Haverford College, West Haverford, Pa., answered all the questions but V. (M. C. STEVENS, Professor.)

CHARLES W. HASSLER, Columbian College, Washington, D. C., answered all the questions but III. and V. (EDWARD T. FRISTOE, Professor.)

HORACE OTIS, Adams Centre, N. Y., answered all the questions but V.

MARQUIS HALL, Brimfield, Mass., answered all the questions but III. and V.

ASHER B. EVANS, Madison University, Hamilton, N. Y., answered all the questions.

JAMES B. FOSSETT, Student in New London Institute, answered all the questions but III. and V. (EPHRAIM KNIGHT, Professor.)

L. E. NEWCOMB, East Machias, Me., answered all the questions but III. and V.

J. C. ELLIOTT, Junior Class, Indiana University, Bloomington, answered questions I. and IV. (DANIEL KIRKWOOD, Professor.)

J. W. JENKS, Senior, Columbia College, New York City, answered all the questions but III. and V. (WILLIAM G. PECK, Prof.)

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